EXPANDED RESEARCH STATEMENT

JOSEPH VAN NAME

1. Introduction

My research interests currently include general topology, point-free topology, ordered sets (especially Boolean algebras and lattices), universal algebra, category theory, and set theory. I have focused especially on connections between these research areas through dualities that resemble Stone duality. In the next three sections, I will describe the three major research projects that I have undertaken within the last two years, namely, my work on Boolean partition algebra, admissibility-systems, and varieties generated by algebras where every function is a fundamental operation.

2. Boolean Partition Algebras

In the paper [22], Stone constructed a duality between Boolean algebras and compact totally disconnected spaces. This duality, known as Stone duality, is the central result in the theory of Boolean algebras since it translates results back and forth between Boolean algebras and general topology. Since Stone proved this result, many people have formulated dualities that generalize Stone duality or are similar to Stone duality including Priestley duality [18] and the duality between spatial locales and sober spaces [17][Ch 2] (See also [12]). In 2010, I came up with the notion of a Boolean partition algebra in order to generalize Stone duality to uniform spaces, and since then I was able to prove results about uniform spaces using Boolean partition algebras. Soon after developing the notion of a Boolean partition algebra, I realized that Boolean partition algebras have much motivation besides the duality between uniform spaces, and they are also applicable to mathematical logic, category theory, and point-free topology. My dissertation is on Boolean partition algebras, and in my dissertation I have focused on the algebraic structure of Boolean partition algebras and their relation to uniform spaces.

A partition of a Boolean algebra $B$ is a subset $p \subseteq B \setminus \{0\}$ such that $\bigvee p = 1$ and if $a, b \in p, a \neq b$, then $a \wedge b = 0$. The collection of all partitions of a Boolean algebra $\mathcal{P}(B)$ is a meet-semilattice partially ordered by $\leq$ (refinement) where $p \leq q$ if for each $a \in p$ there is a $b \in q$ with $a \leq b$. A Boolean partition algebra is a pair $(B, F)$ where $B$ is a Boolean algebra and $F$ is a filter on the meet-semilattice $\mathcal{P}(B)$ such that $B = \{0\} \cup \bigcup F$. For example, if $B$ is a Boolean algebra, then $(B, \mathcal{P}(B))$ is a Boolean partition algebra. Furthermore, if $\lambda$ is an infinite cardinal, then $(B, \{p \in \mathcal{P}(B) : |p| < \lambda\})$ is also a Boolean partition algebra. Boolean partition algebras behave algebraically even though they are second order structures. For instance, one defines $F$-ideals in a Boolean partition algebra $(A, F)$ to be ideals $I$ such that if $p \in F, R \subseteq p \cap I$, and $\bigvee R$ exists then $\bigvee R \in I$. One may take quotients modulo $F$-ideals, and the isomorphism theorems hold for Boolean partition algebras. There are also many ways to construct new Boolean partition algebras from old ones.
including standard constructions such as taking direct products, direct limits, and subalgebras along with other constructions that are unique to Boolean partition algebras.

A uniform space is a pair \((X, \mathcal{U})\) such that \(\mathcal{U}\) is a filter on \(X^2\) that satisfies the following:

1. If \(R \in \mathcal{U}\), then \(1_X \subseteq R\).
2. If \(R \in \mathcal{U}\), then \(R^{-1} = \{(x, y) | (y, x) \in R\} \in \mathcal{U}\).
3. If \(R \in \mathcal{U}\), then there is some \(S \in \mathcal{U}\) with \(S \circ S = \{(x, z) | (x, y) \in S, (y, z) \in S\}\) for some \(y \in X\).

A uniform space is said to be separated if \(\bigcap \mathcal{U} = \{(x, x) | x \in X\}\). For example, if \((X, d)\) is a metric space, then the sets of the form \(\{(x, y) \in X^2 | d(x, y) < \epsilon\}\) generate a uniformity on \(X\). Similarly, if \(X\) is a set and \(D\) is a collection of pseudometrics on \(X\), then one can construct a uniformity from the collection \(D\) of pseudometrics. In fact, any uniform space can be constructed this way. Uniform spaces are topological spaces but with extra structure that gives us the notions of uniform continuity, Cauchy filters, completeness, and uniform convergence. Intuitively, uniform spaces are to uniform continuity as topological spaces are to continuity. Every uniform space has a natural topology, and a topological space is induced by some uniformity if and only if it is completely regular. In other words, uniform spaces always give spaces that satisfy good separation axioms. Uniform spaces are a standard topic in general topology, and they are discussed in some well known classical topology textbooks such as [7] and [23]. Also, the books [10] and [11] focus specifically on uniform spaces.

A uniform space \((X, \mathcal{U})\) is said to be non-Archimedean if \(\mathcal{U}\) is generated by equivalence relations. For example, if \(\mathcal{U}\) is a \(\sigma\)-complete filter, then \((X, \mathcal{U})\) is always a non-Archimedean uniform space. If \((X, \mathcal{U})\) is a uniform space, then the collection of closed subsets \(H(X)\) of \(X\) can be given a natural uniformity, and this uniform space is called the hyperspace of \((X, \mathcal{U})\). If the hyperspace \(H(X)\) is a complete uniform space, then \(X\) is also complete, but there are complete uniform spaces with hyperspaces that are not complete. The uniform space \((X, \mathcal{U})\) is said to be supercomplete if the hyperspace \(H(X)\) is complete. For example, every compact space is supercomplete and every complete metric space is supercomplete.

In order to state Theorem 2.1, we will need the following definitions. A Boolean partition algebra \((B, F)\) is subcomplete if whenever \(p \in F\) and \(R \subseteq p\), then the least upper bound \(\bigvee R\) exists. A Boolean partition algebra \((B, F)\) is stable if each element in \(B \setminus \{1\}\) is contained in a maximal \(F\)-ideal, and \((B, F)\) is superstable if each \(F\)-ideal is extendible to a maximal \(F\)-ideal.

**Theorem 2.1.** 1. The category of complete non-Archimedean uniform spaces is contravariantly equivalent to the category of subcomplete stable Boolean partition algebras.

2. This equivalence restricts to a contravariant equivalence between the full subcategories of supercomplete non-Archimedean uniform spaces and subcomplete superstable Boolean partition algebras.

3. As a consequence, the category of ultraparacompact spaces is contravariantly equivalent to the category of subcomplete superstable locally refinable Boolean partition algebras.

As a special case of this result, one may recover Stone’s original duality between compact totally disconnected spaces and Boolean algebras, and one may recover
the duality found in [19]. One may also use this duality to characterize weakly compact, measurable, and strongly compact cardinals in terms of topological spaces and uniform spaces (compare this with [16] and [1]).

Intuitively, a Boolean partition algebra is essentially a surjective inverse limit of sets. Recall that a downward directed set is a poset set $D$ such that if $d_1, d_2 \in D$, then $d \leq d_1, d \leq d_2$ for some $d \in D$. Let $\text{PS}$ denote the category of all systems $((X_d)_{d \in D}, (\phi_{d,e})_{d,e \in D, d \leq e})$ such that $D$ is a downward directed poset, each $\phi_{d,e}$ is surjective, $\phi_{e,f} \circ \phi_{d,e} = \phi_{d,f}$ for $d \leq e \leq f$, and $\phi_{d,d}$ is the identity function for $d \in D$. The class $\text{PS}$ becomes a category where we define

$$\text{Hom}((X_d)_{d \in D}, (Y_e)_{e \in E}) \equiv \lim_{\longrightarrow, e \in E} \lim_{\longrightarrow, d \in D} \text{Hom}(X_d, Y_e),$$

and the category $\text{PS}$ is a full subcategory of the category of all pro-sets (see [15][Ch. 6] for information on pro-objects). I proved that the category $\text{PS}$ is contravariantly equivalent to the category of subcomplete Boolean partition algebras. The morphisms between Boolean partition algebras $(A, F)$ and $(B, G)$ are much simpler than the morphisms between pro-sets since they are simply Boolean algebra homomorphisms $\phi : A \rightarrow B$ such that $\phi(p) \setminus \{0\} \in G$ for each $p \in F$. Also, subcomplete Boolean partition algebras can be thought of as point-free uniform spaces since the category of subcomplete Boolean partition algebras is equivalent to a subcategory of the category of uniform frames which I call ultracomplete uniform frames. Boolean partition algebras can even be interpreted as Boolean algebras with specified least upper bounds and greatest lower bounds.

Subcomplete Boolean partition algebras may be used in constructing ultrapowers and reduced powers. The ultrapower construction is used to extend any first order structure to a larger first order structure that satisfies the same first order sentences regardless of the number of function or relation symbol. The ultrapower construction is fundamental to non-standard analysis since it gives concrete examples of non-standard models. Also, ultrapowers are used extensively in set theory since many of the large cardinal axioms starting with measurable cardinals can be formulated in terms of certain types of ultrapowers. If $(B, F)$ is a subcompact Boolean partition algebra, and $U$ is an ultrafilter on $B$, then let $U_p = \{R \subseteq p \mid \forall R \in U\}$ for $p \in F$. Then each $U_p$ is an ultrafilter on $p$. We therefore define the Boolean partition algebra ultrapower $A^{(B, F)} / U$ of a first order structure $A$ to be the direct limit of ultrapowers $\lim_{\longrightarrow, p \in F} (A^p / U_p)$. The Boolean partition algebra ultrapower generalizes several ultrapower constructions such as the standard ultrapower construction, the Boolean ultrapower [14], and the limit ultrapower [13]. Furthermore, morphisms between Boolean partition algebras induce elementary embeddings between their corresponding Boolean partition algebra ultrapowers. Also, direct limits of ultrapowers can be naturally represented as Boolean partition algebra ultrapowers. One can even represent an iterated Boolean partition algebra ultrapower as a single Boolean partition algebra ultrapower. In other words, if $(B, F), (C, G)$ are subcomplete Boolean partition algebras and $U \subseteq B, V \subseteq C$ are ultrafilters, then there is a subcomplete Boolean partition algebra $(B, F)^{(C, G)}$ and an ultrafilter $U^V$ on $(B, F)^{(C, G)}$ such that $A^{(B, F)^{(C, G)}} / U^V \cong (A^{(B, F)} / U)^{(C, G)} / V$. One may also represent an ultraproduct of Boolean partition algebra ultrapowers naturally as a single Boolean partition algebra ultrapower.
Naively, one may consider the category $C$ whose objects are simply pairs $(X, A)$ where $X$ is a poset and $A$ is a collection of subsets of $X$ with least upper bounds. An approach like this has been studied in [2]. There is a problem with this approach however. We do not just want to consider any pair $(X, A)$ where $A$ is a collection of subsets of $X$ with least upper bounds. For instance, if $R \subseteq X$ has a greatest element, then we would want to include the set $R$ in $A$. In particular, we would want to include the least upper bound of every singleton. Also, if $R \in A, R \subseteq S$ and $\bigvee R = \bigvee S$, then we would also want to include $S$ in $A$. To illustrate this issue further, consider the pairs $(X, \emptyset)$ and $(X, \{\{x\}|x \in X\})$. These pair are not isomorphic in $C$, but intuitively there is not a substantial difference between the least upper bounds considered in $(X, \emptyset)$ and $(X, \{\{x\}|x \in X\})$. This discussion brings us to the notion of an admissibility system and LUB-systems.

A closure system is a pair $(X, C)$ such that $X$ is a set and $C$ is a collection of subsets of $X$ closed under taking arbitrary intersection including the empty intersection. If $(X, C)$ is a closure system, then let $C^* : P(X) \rightarrow P(X)$ be the mapping where if $R \subseteq X$ then $C^*(R)$ is the smallest element in $C$ containing $R$. If $(X, C)$ is a closure system and $\leq$ is a preordering on $X$, then we say that $(X, C)$ has specialization ordering $\leq$ if each $L \in C$ is a lower set (i.e. if $x \in L$ and $y \leq x$ then $y \in L$) and $\downarrow x = \{y \in X|y \leq x\} \subseteq C$ for each $x \in X$. A closure system $(X, C)$ is $T_0$ if the specialization ordering is partially ordered.

Let $X$ be a poset and let $A$ be the collection of subsets of $X$ with least upper bounds. Let $C_A$ be the collection of all lower sets $L \subseteq X$ such that if $R \in A, R \subseteq L$, then $\bigvee R \in L$. In other words, $C_A$ is the collection of all sets that are closed under going downward and taking least upper bounds in $A$. The closure system $C_A$ is a $T_0$-closure system with specialization ordering $\leq$. If $(X, C)$ is a closure system with specialization ordering $\leq$, then let $A_C$ be the collection of all subsets $R \subseteq X$ such that $\bigvee R \in C^*(R)$. In other words, $A_C$ is the collection of all subsets of $X$ whose least upper bound is reachable by the closure operator $C^*$. We say that a pair $(X, A)$ is an admissibility system if $X$ is a poset and $A$ is a collection of subsets of $X$ with least upper bounds with $A = A_{C_A}$. Intuitively, a pair $(X, A)$ is an admissibility system if whenever $R$ is a subset of $X$ whose least upper bound is reachable by a possibly transfinite process of extending $R$ to a lower set and taking least upper bounds in $A$, then $R \in A$. We say that a $T_0$-closure system $(X, C)$ is said to be a LUB-system if $C = C_{A_C}$. The correspondences $A \mapsto C_A, C \mapsto A_C$ give a Galois connection between closure systems with specialization ordering $(X, \leq)$ and collections of subsets of $(X, \subseteq)$ with least upper bounds. This Galois connection gives a Galois correspondence between LUB-system on $(X, \leq)$ and the admissibility systems on the poset $X$. In particular, if $A$ is a collection of sets with least upper
bounds on a poset $X$, then $(X,C_A)$ is a LUB-system. Similarly, if $(X,C)$ is a $T_0$-closure system with specialization ordering $\leq$, then $(X,A_C)$ is an admissibility system on the poset $(X,\leq)$. The category of LUB-systems with “continuous” maps is isomorphic to the category of admissibility systems with order preserving maps preserving least upper bounds. For example, if $X$ is a join-semilattice, and $A$ is the collection of finite subsets of $X$, then $(X,C_A)$ is a LUB-system and $C_A$ is the collection of all ideals in the join-semilattice $X$. Furthermore, if $X$ is a poset and $A$ is the collection of all directed subsets of $X$, then $C_A$ is the collection of all closed subsets of $X$ in the Scott topology $[8],[9]$ and $(X,C_A)$ is a LUB-system. If $X$ is a poset and $C$ is the collection of all lower sets in $X$, then $(X,C)$ is a LUB-system, and $R \in A_C$ if and only if $R$ has a greatest element. If $(X,A)$ is an admissibility system, $Y \subseteq X$, and $B = \{R \subseteq Y | R \in A, \forall R \in Y\}$, then $(Y,B)$ is also an admissibility system. If $X$ is a complete lattice, then $(X,P(X))$ is an admissibility system.

The following basic result gives an intuitive way of defining an admissibility system.  

**Theorem 3.1.** Let $X$ be a poset, and let $A$ be a collection of subsets of $X$ with least upper bounds. Then $R \in A_C$ if and only if
\begin{itemize}
  \item 1. $\bigvee R$ exists and
  \item 2. whenever $Y$ is a poset and $f : X \rightarrow Y$ is an order preserving map with $f(\bigvee S) = \bigvee f[S]$ for $S \in A$ then $f(\bigvee R) = \bigvee f[R]$.
\end{itemize}

I have given a connection between certain admissibility systems and certain ordered topological spaces. An upper limit space is a poset $X$ along with a topology such that
\begin{itemize}
  \item 1. each set $\downarrow x$ is closed and
  \item 2. $\{\downarrow x \cap U | x \in X, U$ is an open upper set$\}$ is a basis for the $X$.
\end{itemize}

For example, the topology on $\mathbb{R}$ generated by intervals of the form $[a,b]$ is an upper limit space. Every upper limit space is zero-dimensional. In particular, each upper limit space is completely regular and Hausdorff.

**Theorem 3.2.** 1. The category of upper limit spaces is isomorphic to the category of all LUB-systems where the closure system is a topological closure system.
2. Let $(X,C)$ be a LUB-system. Then $C$ is a topological closure system if and only if $\emptyset \notin A_C$ and whenever $R_1 \cup \ldots \cup R_n \in A_C$, then $R_i \in A_C$ and $\bigvee R_i = \bigvee (R_1 \cup \ldots \cup R_n)$ for some $i \in \{1,\ldots,n\}$.

The notion of a LUB-system and of an admissibility system can also be formulated in terms of lattices. A based lattice is a pair $(L,A)$ such that $L$ is a complete lattice and $A \subseteq L$ is a subset such that $x = \bigvee \{a \in A | a \leq x\} = \bigvee \downarrow A x$ for $x \in L$. If $(L,A)$ is a based lattice, then $(A,\{\downarrow A x | x \in L\})$ is a $T_0$-closure system. Furthermore, if $(X,C)$ is a $T_0$-closure system, then $(C,\{\downarrow x | x \in X\})$ is a based lattice. These correspondences form an equivalence between the category of based lattices and $T_0$-closure systems. We therefore say that a based lattice $(L,A)$ is a LUB-based lattice if the corresponding closure system $(A,\{\downarrow A x | x \in L\})$ is a LUB-system. Thus we have defined the category of LUB-based lattices so that it is equivalent to the category of LUB-systems. For example, if $L$ is a complete lattice, then $(L,L)$ is a LUB-based lattice. Furthermore, if $(L,A)$ is a LUB-based lattice and $A \subseteq B \subseteq L$, then $(L,B)$ is also a LUB-based lattice. The LUB-based lattices have a nice characterization in terms of a distributivity property.
Theorem 3.3. Let $(L, A)$ be a based lattice. Then $(L, A)$ is a LUB-based lattice if and only if whenever $M$ is a lower set in $A$ where $R \subseteq M, \sqcup^L R \in A \Rightarrow \sqcup^L R \in M$, then $x \land \sqcup M = \sqcup_{m \in M} (x \land m)$ for each $x \in L$.

LUB-systems and LUB-based lattices have a strong connection with point-free topology. Point-free topology is essentially point-set topology but without any references to points. Instead, in point-free topology one studies lattices called frames that look like the lattice of open sets in a topological space. A frame is a complete lattice that satisfies the distributivity law $\bigvee_{i \in I} (x \land y_i) = x \land \bigvee_{i \in I} y_i$. For example, the open sets of a topological space always form a frame. Most topological concepts have point-free analogues including the definitions of regular, completely regular, normal, compact, paracompact, connected, zero-dimensional, extremally disconnected etc. See [12],[17] for information on frames. If $(L, A)$ is a based lattice and $L$ is a frame, then $(L, A)$ is a LUB-based lattice. Therefore, one is able to represent all frames in terms of admissibility systems and LUB-systems. We say that an admissibility system $(A, \mathcal{A})$ is a localic admissibility system whenever $R \in A$ and $x \leq \bigvee R$, then there is some $S \subseteq \downarrow x = \{ a \in A | a \leq x \}$ where $S \in \mathcal{A}, \bigvee S = x$ and where for each $s \in S$ there is an $r \in R$ with $s \leq r$. The category of all localic admissibility systems is equivalent to the category of all based lattices $(L, A)$ such that $L$ is a frame. Localic admissibility systems can be described in terms of Grothendieck topologies on posets (see also [20]).

I gave special attention to the relation between zero-dimensional frames and admissibility system since this correspondence allows one to represent some frames that satisfy good separation axioms in terms of admissibility systems. If $L$ is a frame, then let $\mathfrak{B}(L)$ denote the Boolean algebra of complemented elements in $L$. In other words, $x \in \mathfrak{B}(L)$ if and only if there is some $y \in L$ with $x \land y = 0$ and $x \lor y = 1$. We say that $L$ is a zero-dimensional frame if $(L, \mathfrak{B}(L))$ is a based lattice. The notion of a zero-dimensional frame is clearly the point-free analogue of a zero-dimensional space. A Boolean admissibility system is a localic admissibility system $(B, \mathcal{A})$ such that $B$ is a Boolean algebra and $\mathcal{A}$ contains each finite subset of $B$, and a Boolean admissibility system $(B, \mathcal{A})$ is said to be subcomplete if whenever $R, S \in \mathcal{A}$ and $r \land s = 0$ for $r \in R, s \in S$, then $R \in \mathcal{A}$ as well. The category of Boolean admissibility systems is equivalent to the category of all pairs $(L, A)$ such that $L$ is a zero-dimensional frame and $A$ is a Boolean subalgebra of $L$. With this duality between zero-dimensional frames and Boolean admissibility systems I was able to characterize several topological properties of zero-dimensional frames in terms of admissibility systems. These properties include ultranormality, ultraparacompactness, $\kappa$-compactness, extremally disconnected zero-dimensional frames(as complete Boolean admissibility systems), Lindelof $P$-frames (as $\sigma$-complete Boolean algebras), and other properties. Also, by representing zero-dimensional frames as admissibility systems over Boolean algebras, I was able to characterize the zero-dimensional spaces in terms of admissibility systems over Boolean algebras that satisfy very strong distributivity properties similar to those given in [4].

4. Varieties and Pro-Objects

If $A$ is an infinite set, then let $\Omega(A)$ be the algebra with underlying set $A$ where every function from $A^n$ to $A$ is a fundamental operation and each element of $A$ is a fundamental constant. In other words, $\Omega(A)$ is an infinite primal algebra. Let $V(\Omega(A))$ denote the variety generated by $\Omega(A)$. I originally began investigating the
varieties of the form $V(\Omega(A))$ in the summer of 2012 in order to further understand ultrapowers and reduced powers. I have not found works in the current mathematical literature that study infinite primal algebras in considerable detail yet. The paper [21] is the only work that I have found that studies infinite primal algebras and this paper only covers a few very basic properties of infinite primal algebras.

The variety $V(\Omega(A))$ has very strong algebraic properties since the algebra $\Omega(A)$ has all possible fundamental operations. For instance, if $\theta$ is a congruence on some power $\Omega(A)^I$, then there is some filter $Z$ on $I$ such that $(f, g) \in \theta$ if and only if \{ $i \in I | f(i) = g(i)$ $\} \in Z$. Furthermore, if $B$ is a subalgebra of $\Omega(A)^I$, then there is some filter $F$ on the lattice of equivalence relations on $I$ such that $f \in B$ if and only if $E_f \in F$ where $E_f$ is the equivalence relation on $I$ induced by the function $f$. Also, the algebra $\Omega(A)^{A^n}$ is freely generated by the projections $\pi_i : A^n \to A$ onto the $i$-th coordinate for $i = 1, ..., n$. More generally, if $A$ is given the discrete uniformity and $A^I$ is given the product uniformity, then if $F(A, I)$ is the set of uniformly continuous functions from $A^I$ to $A$, then $\mathcal{F}(A, I)$ is a free algebra in $V(\Omega(A))$ freely generated by the projections from $A^I$ to $A$.

Using slight generalizations of these very strong properties of the variety $V(\Omega(A))$ and Birkhoff’s theorem, one can give a proof of Keisler’s theorem [13][3.7] which states that every complete elementary embedding is a limit ultrapower (see also [6] for information concerning limit ultrapowers). Moreover, from an elementary superstructure $\mathcal{L}$ of $\Omega(A)$, I can construct a specific limit ultrapower of $\Omega(A)$ isomorphic to $\mathcal{L}$ simply by representing $\mathcal{L}$ as a quotient of a free algebra. Furthermore, if $\mathcal{L} \in V(\Omega(A))$, then one can represent $\mathcal{L}$ as a limit reduced power of $\Omega(A)$. In other words, the algebras in $V(\Omega(A))$ are essentially limit reduced powers or even limit ultrapowers. The homomorphisms between algebras in $V(\Omega(A))$ generally can be represented in terms of uniformly continuous functions.

The model-theoretic structure of $V(\Omega(A))$ seems to be very interesting and very promising. For instance, the first order theory of $V(\Omega(A))$ is generated by the identities in $V(\Omega(A))$ and a single sentence such as $\forall x, y, z(x \neq y \rightarrow f(x, y, z) = x)$ where $f : A^3 \to A$ is any function where $a \neq b \rightarrow f(a, b, a) = a$ and $f(a, a, b) = b$ (such a function $f$ is called a discriminator function [3][Sec. 4.9]). It will therefore probably be worthwhile to study the connection between the varieties $V(\Omega(A))$ and model theory in the future.

The varieties of the form $V(\Omega(A))$ can be used to form the following duality. Let $\mathfrak{S}$ be the second category discussed in [3] (in this category the morphisms are the equivalence classes of functions). In this category, the objects are pairs $(X, \mathcal{F})$ such that $X$ is a set and $\mathcal{F}$ is a filter on $X$. Let $PF$ be the category of inverse systems $(X_d, \mathcal{F}_d)_{d \in D}$ in $\mathfrak{S}$ over downward directed sets $D$ where each transitional mapping is an epimorphism. The category $PF$ is a full subcategory of $Pro(\mathfrak{S})$ where $Pro(\mathfrak{S})$ denotes the category of all inverse systems in $\mathfrak{S}$. If $\kappa$ is a cardinal, then let $PF_\kappa$ be the subcategory of $PF$ consisting of inverse systems $(X_d, \mathcal{F}_d)_{d \in D}$ such that $|X_d| \leq \kappa$ for $d \in D$. The category $PF$ is very rich since this category contains all filters, subcomplete Boolean partition algebras, and complete non-Archimedean uniform spaces as full subcategories up to equivalence and contravariant equivalence.

**Theorem 4.1.** For an infinite set $A$, the category $PF_{|A|}$ is contravariantly equivalent to the variety $V(\Omega(A))$.

Take note that since $PF$ is the union of the categories $PF_\kappa$, we may approximate the category $PF$ by the subcategories $PF_\kappa$. Therefore, one may represent any
element in the category $\mathbf{PF}$ by algebras in some $V(\Omega(A))$ for sufficiently large sets $A$.

I conjecture that the category $\mathbf{PF}$ is equivalent to the category of pro-sets. I also conjecture that the category $V(\Omega(A))$ is contravariantly equivalent to $\text{Pro}(\text{Set}_{|A|})$ where $\text{Set}_{|A|}$ is the category of sets of cardinality at most $|A|$. The proofs of these conjectures should be fairly straightforward, but I have not had time to prove these conjectures yet. If these conjectures hold, then the rather complicated morphisms between pro-sets reduce to homomorphisms between algebras in varieties of the form $V(\Omega(A))$ and also equivalence classes of uniformly continuous mappings between certain uniform spaces. Furthermore, if these conjectures are correct, then we obtain a new connection between category theory and set theory.

5. Other Work

Besides my work described in the above sections, I have done a couple of other research projects as a graduate student as well over the past two years.

For instance, I have done some research in proximity spaces. Proximities are analogous to topological spaces in that while topological spaces have a notion of whether a point is arbitrarily close to a set, proximity spaces have a notion whether a set is arbitrarily close to another set or whether those two sets are separated. Therefore one defines a proximity space as a pair $(X, \delta)$ where $\delta$ is a binary relation on the powerset $P(X)$ where $A \delta B$ if the sets $A$ and $B$ are “touching” each other in some sense. Every proximity space $(X, \delta)$ induces a topology on the set $X$ and this topology is always completely regular. In fact, every completely regular space is induced by some proximity. Moreover, the proximities on a completely regular space are in a one-to-one correspondence with the compactifications of that completely regular space.

I wrote a paper that recently got accepted on proximity spaces, and in this paper I generalized the notion of a $P$-space to proximity spaces. I therefore defined a $P$-proximity to be a proximity space such that if $A \delta \bigcup_{n=1}^{\infty} B_n$, then $A \delta B_n$ for some $n$. In this paper, I showed that the category of $P$-proximities is isomorphic to the category of $\sigma$-algebras. Furthermore, I showed that the $P$-proximity coreflection of a proximity space corresponds to the $\sigma$-algebra of proximally Baire sets (i.e. the $\sigma$-algebra generated by the zero sets of proximity maps into $\mathbb{R}$). As a corollary, I also proved the proximity maps from a $P$-proximity space into $\mathbb{R}$ are precisely the measurable functions.

I also have studied generalizations of the notions of realcompactness and pseudocompactness to proximity spaces, and I have given several non-trivial characterizations of the proximally realcompact spaces and proximally pseudocompact spaces. For example, I gave a one-to-one correspondence between the points of the proximal realcompactification of a proximity space and the $\sigma$-complete ultrafilters on the $\sigma$-algebra of proximally Baire sets. From this characterization of the proximal realcompactification, I concluded that a proximity space $(X, \delta)$ is proximally realcompact if and only if every $\sigma$-complete ultrafilter on the $\sigma$-algebra of proximally Baire sets is a principal ultrafilter. I also gave a couple functional analytic characterizations of proximally pseudocompact spaces.

I also did some work relating topological and uniform spaces to large cardinals. I am also familiar with different areas of mathematical analysis since about half of the graduate level classes that I have taken are in analysis. As a summer 2012 work
project, I have been exposed to research areas such as potential theory by Vilmos Totik.

6. Future Plans

In the near future, I will need to continue to investigate pro-completions and the varieties $V(\Omega(A))$ since this is my newest major research project and I have not had time to thoroughly investigate this topic yet. I also want to study algebras that are similar to $V(\Omega(A))$ like discriminator varieties. I also may consider investigating a generalization of the notion of an admissibility system where one considers both least upper bounds and greatest lower bounds. I also want to learn more about areas related to set theory especially large cardinals since this subject is very interesting and it is related to several aspects of my research. I will be open to learning and researching most areas of pure mathematics, and I am eager to collaborate with other mathematicians and further expand my research areas.

References


E-mail address: jvanname@mail.usf.edu